Statistical Power for Logistic Regression

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Outline

- Introduction
  - Power
  - Logistic regression
- Hypothesis testing
  - Wald test
  - Score test
  - Likelihood ratio test
- Power calculation
- WebPower and future directions

Why power?

- Power: the probability of rejecting the null hypothesis given that the null hypothesis is false.

$$\text{power} = P(\text{reject null} | \text{null is false})$$

- two forms of power analysis are most useful:
  - the determination of the sample size n required to attain a specified degree of power
  - the determination of the power to detect the hypothesized effect size.

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4. Others: the tests methods, distribution of predictors, missing data ....

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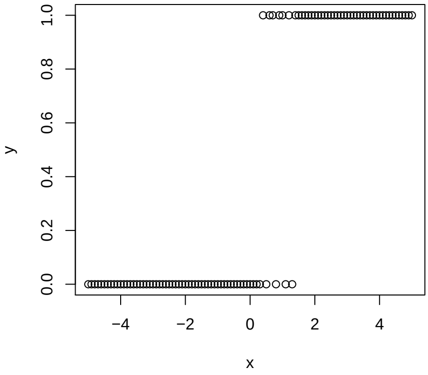

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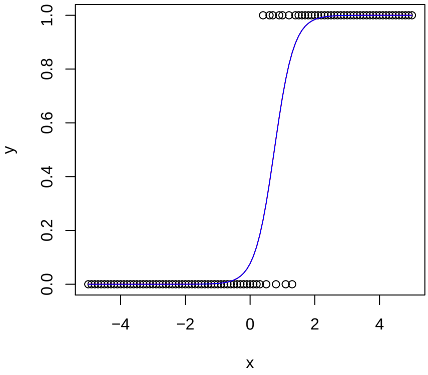

Estimation

- Likelihood function/log-likelihood

$$
\begin{aligned}
L(b_0, b_1, \cdots, b_p; y, x) &= \Pi_{i=1}^{N} \pi(x_i)^{y_i} [1 - \pi(x_i)]^{1-y_i} \\
l(b_0, b_1, \cdots, b_p; y, x) &= \sum_{i=1}^{N} \{ y_i \pi(x_i) + (1 - y_i)[1 - \pi(x_i)] \}
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- Newton-Raphson method: suppose $\hat{\beta}$ is MLE of β, when $n \rightarrow \infty$
  1. $\hat{\beta} \sim N(\beta', \Sigma)$(asymptotically normal)
  2. $\beta' = \beta$(consistency)
  3. $\Sigma = I^{-1}$ where I is the Fisher-information matrix. $I_{jk} = -E(\frac{\partial^2 l(\beta)}{\partial \beta_j \beta_k})$. Particularly, in one dimension, $\sigma^2 = cov(\hat{\beta}) = I^{-1} = E[-\frac{\partial^2 l(\beta)}{\partial \beta^2}]$.

Hypothesis testing

In simple logistic regression, let $\beta = b_1$(slope)

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- Effect size index:

$$b_1 = \log \frac{p_1/(1-p_1)}{p_0/(1-p_0)}$$

where $p_0 = Pr(Y = 1|H_0)$ and $p_1 = Pr(Y = 1|H_1 \& X = 1)$.

- Contional on H_1 is true, but we fit the H_0, we get

$$b_0^* = \log \frac{\pi}{1-\pi}$$

where $\pi = E(\pi(x))|_{H_1}) = E_{X\sim F}(\frac{1}{1+\exp(-b_0-b_1X)})$.

Three test statistics for $H_0 : \beta = 0$

- Wald test: $Z_w = \dfrac{\hat{\beta}}{\sqrt{var(\hat{\beta})|_{\beta=\hat{\beta}}}} \rightsquigarrow N(0,1)$
- Score test : $Z_s = \dfrac{\frac{\partial l(\beta)}{\partial \beta}|_{\beta=0}}{\sqrt{-E(\frac{\partial^2 l(\beta)}{\partial \beta^2})|_{\beta=0}}} \rightsquigarrow N(0,1)$
- Likelihood ratio test: $S_{LR} = 2(l_1 - l_0) \rightsquigarrow \mathcal{X}^2_{df=1}$

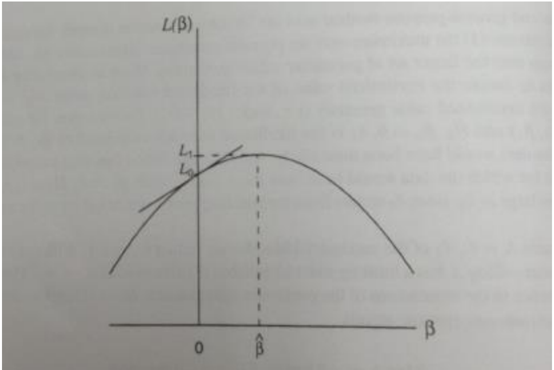

Figure 1.1 Log-likelihood function and information used in three tests of H_0: $\beta = 0$.

Power Calculation

Hypothesis: $H_0 : \beta = 0$ vs $H_1 : \beta \neq 0$:

$$\text{Power} = P(H_0 \text{ is rejected} | H_1 \text{ is true})$$

- Under H_0 :

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- Bickel, Doksum(2001), Demidenko(2004), by Slusky theorem

$$var(\hat{\beta})|_{\hat{\beta}} \to var(\hat{\beta})|_{\beta}$$

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What's $var(\hat{\beta})|_{\beta=\beta}$?

- Log-likelihood

$$l(b_0, b_1; y, x) = \sum_{i=1}^{N} y_i \log \pi(x_i) + (1 - y_i) \log(1 - \pi(x_i))$$

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- Fisher-information:

$$\frac{\partial l_i(b_0, b_1)}{\partial b_0^2} = -\pi(x_i)[1 - \pi(x_i)]$$

$$\frac{\partial l_i^2(b_0, b_1)}{\partial b_1^2} = -x_i^2 \pi(x_i)(1 - \pi(x_i))$$

$$\frac{\partial l_i^2(b_0, b_1)}{\partial b_0 b_1} = -x_i \pi(x_i)[1 - \pi(x_i))]$$

$$I = n \begin{bmatrix} E\pi(x_i)[1 - \pi(x_i)] & Ex_i\pi(x_i)(1 - \pi(x_i)) \\ E\pi(x_i)[1 - \pi(x_i)] & Ex_i^2\pi(x_i)(1 - \pi(x_i)) \end{bmatrix} = \begin{bmatrix} I_{00} \\ I_{01} \end{bmatrix}$$

$$I^{-1} = \frac{1}{n(I_{00} I_{11} - I_{01}^2)} \begin{bmatrix} I_{11} & -I_{01} \\ -I_{01} & I_{00} \end{bmatrix}$$

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\end{aligned}$$

- $var(\hat{\beta}) = \frac{I_{00}}{n(I_{00} I_{11} - I_{01}^2)} = v_1/n$ in which b_0 and b_1 are given by H_1

Variance with correction

- $var(\hat{\beta}) = \frac{I_{00}}{n(I_{00}I_{11}-I_{01}^2)} = v_1/n$ in which b_0 and b_1 are given by H_1.
- Under H_1 :

$$\Gamma = \frac{\hat{\beta}-\beta}{\sqrt{\frac{v_1}{n}}} \rightsquigarrow N(0,1)$$

- With variance correction,

$$\Gamma = \frac{\sqrt{n}(\hat{\beta}-\beta)}{\sqrt{v_1}} \rightsquigarrow N(0, \sqrt{\frac{av_0+(1-a)v_1}{v_1}})$$

where $v_0$is the variance of $\hat{\beta}_1$for H_0 with $b_0 = \lg \frac{\mu}{1-\mu}$, with

$$\mu = E_X(\frac{\exp(b_0+b_1X)}{1+\exp(b_0+b_1X)})$$

- Thus under the $H_1 : \Gamma = \frac{\hat{\beta}-\beta}{\sqrt{\frac{av_0+(1-a)v_1}{n}}} \sim N(0,1)$
- Power$= \Phi(-Z_{1-\frac{\alpha}{2}} + \sqrt{n}\beta/\sqrt{av_0+(1-a)v_1}) + \Phi(-Z_{1-\frac{\alpha}{2}} - \sqrt{n}\beta/\sqrt{av_0+(1-a)v_1})$.

Variances

In the power expression:

1. $v_0 = \frac{I_{00}}{I_{00} I_{11} - I_{01}^2}$ evaluated under $H_0 : b_{0*} = \log$
   1.1 and $v_1 = \frac{I_{00}}{I_{00} I_{11} - I_{01}^2}$ evaluated under H_1
2. Under H_1 :
   2.1 $I_{00} = E\pi(x)[1 - \pi(x)]\backslash$
   2.2 $I_{11} = Ex^2\pi(x)[1 - \pi(x)]$
   2.3 $I_{01} = Ex\pi(x)[1 - \pi(x)]$
3. $X \sim Bernoulli(B)$

$$I_{00} =$$

Distribution of the covariate

- Bernoulli(p)
- Exponential(λ)
- Lognormal(μ, σ)
- Normal(μ, σ)
- Poisson(λ)
- Uniform(L, R)

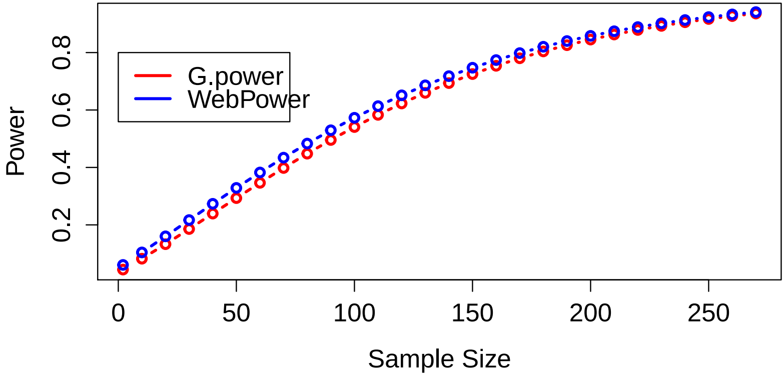
WebPower

- Input model:
  - $p_0 = Pr(Y = 1|H_0)$ and $p_1 = Pr(Y = 1|H_A, X = 1)$
  - significant level α
  - either n or power
  - family: distribution of the predictor

`http://webpower.psychstat.org/`

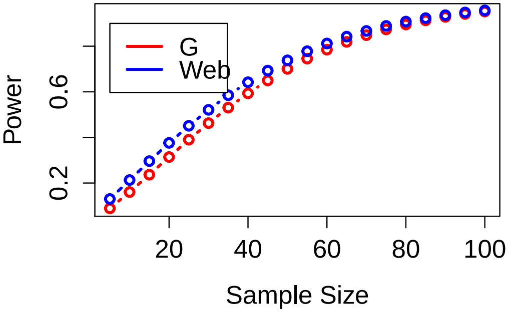
WebPower VS G-power

$p_0 = 0.2, p_1 = 0.3, \alpha = 0.05,\ X \sim N(0,1)$



WebPower VS G-power

$p_0 = 0.2, p_1 = 0.3, \alpha = 0.05,\ X \sim U[0, 5]$



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Acknowledgment

- Johnny Zhang
- Ke-Hai Yuan
- My colleagues
  - Yujiao, Megan, Han, Ge, Miao, Asangus